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Reduction of quantization error in position encoders

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Abstract. The paper describes estimate of the quantization error of a real position encoder taking into account the factors specified above, and the method of minimization of this error by an additional adjustment of the scale is proposed.

Using position encoders in digital indication or *CNC* systems of machine tools [1–4] leads to some amplitude quantization error due to the fact that the transformed function is fixed as a constant within quanta of the sensor scale [5–7]. This error, within a quantum, conforms to the law of uniform distribution and, at the quantum value λ, is characterized by mathematical expectation

$$M(\Delta\lambda)=\frac{1}{\lambda}\int_0^{\lambda}xdx=\frac{\lambda}{2}$$

and dispersion

$$D(\Delta\lambda)=\frac{1}{\lambda}\int_0^{\lambda}\left(x-\frac{\lambda}{2}\right)^2dx=\frac{\lambda^2}{12}$$

If the encoder scale is adjusted symmetrically with regard to $M(\Delta\lambda)$ (in which case the centers of the adjustment coincide with the centers of quantum symmetry), then the systematic component of the quantization error can be excluded. Based on this, the quantization errors of position encoders are usually evaluated with $D(\Delta\lambda)$or $\pm\sqrt{D(\Delta\lambda)}$ [8–10]. However, such estimate is idealized, because it does not consider the instrumental errors (manufacturing and assembly errors) of the encoder [11, 12] and the distribution law of the transformed parameter [13]. Below, an attempt is made to estimate the quantization error of a real position encoder taking into account the factors specified above, and the method of minimization of this error by an additional adjustment of the scale is proposed.

Instrumental errors cause deviation of real quantum values from theoretical ones and lead to errors of their location. In this regard, symmetric adjustment and exclusion of the systematic component of the quantization error in the real encoder are possible only for one quantum. For all others, it is practically infeasible.

Let us find the systematic component of the quantization error for a *k*-th scale quantum, assuming that the scale is adjusted in the middle of the first quantum. Let the scale of the encoder have *p* quanta, the actual values of which are $a_1, a_2, ..., a_k, ..., a_p$ and the theoretical ones – $b_1, b_2, ..., b_k, ..., b_p$. Then the distances from the alignment centers to the right limits of the quanta on the real scale (see figure 1 below) are equal to:

for the first quantum $-c_1 = a_1/2$;

for the second one $-c_2 = \frac{a_1}{2} + a_2 - \frac{b_1}{2} - \frac{b_2}{2}$;

for the third one $-c_3 = \frac{a_1}{2} + a_2 + a_3 - \frac{b_1}{2} - b_2 - \frac{b_3}{2}$;

and for the k-th one $-$ $c_k = \frac{a_1}{2} + \frac{1}{2}\left[1 - \sum_{r|k} \mu(r)\right](2a_k - b_1 - b_k) + \sum_{i=2}^{k-1}(a_i - b_i)$,

where $\mu(r)$ – Mobius function [14].

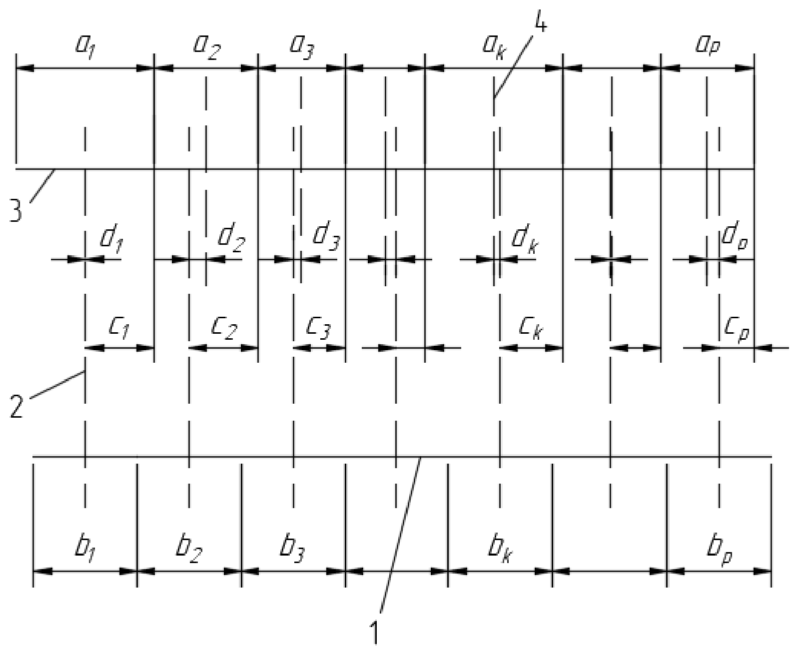

Figure 1. Scheme for determining the systematic component of the quantization error: 1 – the scale of an ideal encoder; 2 – quanta alignment centers; 3 – the scale of the encoder; 4 – centers of quanta symmetry of a real scale.

On the other hand, as can be seen from the figure, $c_k = \frac{a_k}{2} - d_k$, where d_k is the systematic component of the quantization error within the k-th quantum. It follows that

$$d_k = \frac{1}{2}\left\{a_k - a_1 - \left[1 - \sum_{r|k} \mu(r)\right](2a_k - b_1 - b_k)\right\} - \sum_{i=2}^{k-1}(a_i - b_i). \quad (1)$$

Instrumental errors cause deviation of real quantum values [12, 15] from theoretical d_k and the variance of the quantization error within the k-th quantum $D(\Delta a_k) = \frac{a_k^2}{12}$ it is easy to find the most probable limit values of the total quantization error for the k-th quantum of the encoder scale:

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$$f_k = \frac{1}{2}\left\{a_k - a_1 - \left[1 - \sum_{r|k}\mu(r)\right](2a_k - b_1 - b_k)\right\} - \sum_{i=2}^{k-1}(a_i - b_i) \pm \frac{a_k}{2\sqrt{3}}. \tag{2}$$

If we take now into account that f_k (for k = 1, 2, ..., p) is a random variable, the probability of whose occurrence is equal to P_k (i.e. the probability that the quantized function being in the k-th interval), we can find the mathematical expectation and dispersion of the quantization error for the entire encoder:

$$\mathrm{M}(f) = \sum_{k=1}^{p} P_k f_k; \tag{3}$$

$$D(f) = \sum_{k=1}^{p}[f_k - M(f)]^2 P_k. \tag{4}$$

Each value of k has two corresponding values of f_k – positive and negative. Therefore, calculation of $M(f)$ and $D(f)$ should be carried out separately for those and other f_k, considering further such calculated values which are greater by their absolute value (modulus) to be the desired ones. The analysis shows that the found value $|M(f)|$ can be minimized without increasing $D(f)$. For that purpose, besides the adjustment of the encoder scale in the middle of one of the quanta, it is sufficient to make an additional adjustment by the value

$$M(d) = \sum_{k=1}^{p} P_k d_k. \tag{5}$$

After that additional adjustment we obtain $M(f) = M(f')$ and $D(f) = D(f')$, where $f'_k = f_k - M(d)$. As

$$M(f') = \sum_{k=1}^{p} P_k[f_k - M(d)] = \sum_{k=1}^{p} P_k d_k \pm \frac{1}{2\sqrt{3}}\sum_{k=1}^{p} P_k a_k - \sum_{k=1}^{p} P_k \sum_{k=1}^{p} P_k d_k \,,$$

$$\text{where } \sum_{k=1}^{p} P_k \sum_{k=1}^{p} P_k d_k = \sum_{k=1}^{p} P_k d_k \sum_{k=1}^{p} P_k = \sum_{k=1}^{p} P_k d_k \,,$$

$$\text{then } M(f') = \pm\frac{1}{2\sqrt{3}}\sum_{k=1}^{p} P_k a_k. \tag{6}$$

Equal moduli of $M(f')$ positive and negative values indicate that $|M(f)| = \min$, since, with the additional adjustment of the scale, the modulus of a negative $M(f)$ can be only reduced by increasing the modulus of a positive $M(f)$, and vice versa.

Returning to $D(f')$, one can show that

$$D(f') = \sum_{k=1}^{p}[f_k - M(d) - M(f')]^2 P_k = \sum_{k=1}^{p}\left(d_k \pm \frac{a_k}{2\sqrt{3}} - \sum_{k=1}^{p} P_k d_k \mp \frac{1}{2\sqrt{3}}\sum_{k=1}^{p} P_k a_k\right)^2 P_k \,.$$

However, before the additional adjustment

$$D(f) = \sum_{k=1}^{p}[f_k - M(f)]^2 P_k = \sum_{k=1}^{p}\left(d_k \pm \frac{a_k}{2\sqrt{3}} - \sum_{k=1}^{p} P_k d_k \mp \frac{1}{2\sqrt{3}}\sum_{k=1}^{p} P_k a_k\right)^2 P_k \,.$$

Thus, $D(f') = D(f)$ before and after the additional adjustment.

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Uniform quantization (i.e. with $a = const$) and absolutely precise manufacturing of the encoder scale give us

$$M(f') = \pm\frac{a}{2\sqrt{3}} \sum_{k=1}^{p} P_k = \pm\frac{a}{2\sqrt{3}} \text{ and } D(f') = 0\,.$$

This is quite consistent with the known idealized estimation of quantization errors.

Consider the use of the obtained dependencies in the example of a single-digit section of a multi-section decimal position encoder (with a circular code scales), used for digital indication of movements of the executive part of the machine tool. Suppose that the actual quantum values of the scale of the considered section are $a_1, a_2, ..., a_k, ..., a_{10}$ (see table 1); theoretical ones are $b_1 = b_2 = ...b_{10} = 36°$. The values of d_k and f_k are also given in the table. They are calculated on the basis of that data according to equations (1) and (2) for the case of adjusting the section scale in the middle of the first quantum.

Table 1. Dates of the calculating for the example.

№ of a quantum of the scale	a_k	d_k	f_k	P_k
1	36°.20	0°.000	–10°.462 +10°.462	0.403
2	35°.70	+0°.050	–10°.368 +10°.268	0.091
3	36°.45	–0°.025	–10°.510 +10°.560	0.107
4	35°.25	+0°.125	–10°.313 +10°.063	0.061
5	36°.25	+0°.375	–10°.852 +10°.102	0.076
6	36°.15	–1°.025	–9°.423 +11°.473	0.103
7	36°.50	–1°.350	–9°.199 +11°.899	0.057
8	35°.20	–1°.200	–9°.943 +11°.343	0.036
9	36°.20	–0°.900	–9°.562 +11°.362	0.048
10	36°.10	–1°.050	–9°.383 +11°.483	0.018

Further, we consider that the movements of the executive part of the machine tool to be a function. The probability of this function being within the limits of a certain quantum of the section scale approaches, with increase the number of parts processed on the machine tool, to the probabilities of occurrence of certain numbers in the engineering data. We assume the values of these probabilities to be P_k [16] – they are also given in the table. Knowing P_k and f_k, let us calculate statistical characteristics of quantization errors using equations (3) and (4); for negative errors $M(f) = -10°.181$ and $D(f) = 0°.228$; for positive ones $M(f) = 10°.682$ and $D(f) = 0°.282$.

The maximum achievable accuracy of the considered section in the general case (without the additional adjustment of the scale) can be evaluated for values of $M(f) = 10°.682$ and $D(f) = 0°.282$. If we do make the additional adjustment of the scale by the value of $M(d) = -0°.250$ (calculated by equation (5)), then the value of $M(f') = \pm 10°.431$ (calculated by equation (6)) should be taken as an evaluation of the mathematical expectation of the quantization error within the section.

This example shows that the equations given in the paper allow to evaluate and minimize the statistical characteristics of quantization errors in position encoders quite easily, at the same time taking the instrumental error of the sensor and the law of distribution of the transformed parameter into account.

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